P.G. Sem - III, Functional Analysis

Q - Construct a Banach space of continuous function which is not a Hilbert space.

Answer - Consider the Banach space $C[0,1]$ of all continuous function on the closed interval $[0,1]$ of R with the norm defined by

$$\|f\| = \sup_{t \in [0,1]} |f(t)| \text{ for } f \in C[0,1]$$

This norm does not satisfy the parallelogram law. This can be seen taking $f(t) = t$ and $g(t) = 1-t$ for $t \in [0,1]$.

Then $$\|f\| = 1$$
$$= \|g\|$$
$$= \|f+g\|$$
$$= \|f-g\|$$

Thus,

$$\|f+g\|^2 + \|f-g\|^2 = 1^2 + 1^2 = 2$$

while

$$2\left[\|f\|^2 + \|g\|^2\right] = 2\left[1^2+1^2\right]$$
$$= 2 \times 2 = 4$$

Hence the parallelogram law is not satisfied for the sup norm on $C[0,1]$. But in a Hilbert space the parallelogram law must be satisfied. Therefore, the Banach space $C[0,1]$ with sup norm is a Banach space but not a Hilbert space.